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NUMERICAL SOLUTION OF THE PROBLEM OF MOTION OF A LOAD ON A CRACKED ICE SHEET

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A mathematical model is constructed for the motion of a load on a cracked ice sheet. Examples of calculation of ice deflections are given, and the calculation results are analyzed.

Key words: *ice sheet, crack, moving load.*

The necessity of ice breaking arises in navigation and in operation of engineering facilities in river and offshore water areas. In many cases, this can be effectively done using air-cushion vehicles (ACV). Ice breaking in this case results from both the action of the weight of the vessel moving on the ice surface and the generation of a flexural-gravity wave in ice with an amplitude sufficient for breaking [1]. To develop recommendations on the mode of motion of ACVs during ice breaking, it is necessary to calculate the stress–strain state of the ice sheet.

Observations of ice breaking by a moving ACV have shown that complete breaking is preceded by the occurrence of cracks which dissect ice into large fragments. Because, for some time, ice retains certain load-carrying ability due to the interaction of the cracks faces, in order to determine the parameters of motion of an ACV during ice breaking, one has to calculate the stress–strain state not only of the continuous ice sheet but also of the ice sheet with the cracks formed in it.

To construct a mathematical model of the problem posed, we use rectangular coordinates with the x and y axes in the plane of the ice sheet and the z axis directed upward. According to [2], the ice sheet is represented as a plate and water is considered an ideal incompressible fluid. The depth of the basin is considered constant.

As the basic relations, we use the differential equation of viscoelastic oscillations of the ice sheet [2]

$$D\Big(1+\tau_f\,\frac{\partial}{\partial t}\Big)\nabla^4 w+\rho_w g w+\rho_i h\,\frac{\partial^2 w}{\partial t^2}+\rho_w\,\frac{\partial\Phi}{\partial t}\Big|_{z=0}=p(x,y,t), \tag{1}$$

the Laplace equation for the fluid flow velocity potential

$$\nabla^2\Phi=0, \tag{2}$$

and the boundary conditions at the bottom of the basin

$$\frac{\partial\Phi}{\partial z}\Big|_{z=-H}=0 \tag{3}$$

and at the ice–water interface

$$\frac{\partial w}{\partial t}-\frac{\partial\Phi}{\partial z}\Big|_{z=0}=0. \tag{4}$$

Here w is the ice deflection, t is time, ρ_w and ρ_i are the densities of water and ice, respectively, g is the acceleration due to gravity, h is the thickness of the ice sheet, Φ is the velocity potential, p is the external loading intensity, H is the depth of the basin, τ_f is the strain relaxation time, $D=Eh^3/[12(1-\nu^2)]$ is the flexural stiffness of the plate, E is Young's modulus, and ν is Poisson's constant.

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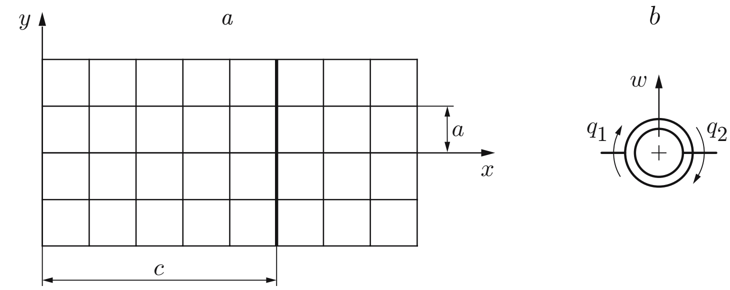

Fig. 1. Discretization of a cracked plate: (a) discrete model; (b) hinged element connecting the crack faces; the bold line is the crack.

The boundary conditions on the crack are specified as follows. During the crack opening, the crack faces are not displaced on the vertical relative to each other, and the forces of their interaction are perpendicular to the faces and are reduced to the distributed moment load, which is the bending moment on the crack face [3]. In this case, at a given point of the crack, the intensity of the moment load m depends on the crack opening angle φ: $k\varphi = m$ (k is the stiffness per unit length, which is identical for all points of the crack). With the adopted assumptions, at a given point of the crack on each of the crack faces, we have the boundary conditions

$$M_\xi = k\Big(\frac{\partial w}{\partial \xi}\Big|_{x-0} - \frac{\partial w}{\partial \xi}\Big|_{x+0}\Big), \qquad M_\xi = -D\Big(\frac{\partial^2}{\partial \xi^2}\Big(w + \tau_f\,\frac{\partial w}{\partial t}\Big) + \nu\,\frac{\partial^2}{\partial \eta^2}\Big(w + \tau_f\,\frac{\partial w}{\partial t}\Big)\Big),$$

where x is the abscissa of the given point in the general coordinate system, w is the deflection at this point, $\partial w/\partial t$ is the rate of change of the deflection at this point, ξ and η are the normal and tangent, respectively, to the crack line at this point in the plane xOy, and M_ξ is the bending moment on the crack face.

This problem is solved numerically using a combination of the finite-element method and the finite-difference method [4].

In the calculation, we consider a horizontally bounded ice region under which there is a fluid. The dimensions of this region should be sufficient to assume no plate displacement on the boundary Γ and to use rigid clamping conditions on Γ. On the vertical surface bounding the fluid under the ice sheet, it is necessary to specify the nonpenetration condition

$$\frac{\partial \Phi}{\partial n}\Big|_{(x,y)\in\Gamma} = 0, \qquad (5)$$

where n is the normal perpendicular to the z axis.

Eliminating the velocity potentials Φ from Eqs. (1)–(5) and performing transformations using the finite-element method [4, 5], we obtain a system of second-order matrix differential equations (see [4, Eq. (10)] or [5, Eq. (2.33)]). Using the finite-difference method to solve this system, we obtain a system of matrix equations (see [4, Eq. (12)] or [5, Eq. (2.35)]), whose solution yields the deflections of the ice plate. In this case, accounting for the boundary conditions on the crack without changing the form of the matrix differential equations leads to a change in only the matrix coefficients.

The initial conditions are formulated for each particular problem. For example, if prior to the beginning of motion, an ACV is at rest at a certain point of the ice sheet, the, at $t = 0$, the plate deflection is equal to the static deflection under the weight of the vessel and the initial velocity is $(\partial w/\partial t)|_{t=0} = 0$. If a load is applied to the ice sheet instantaneously, without a shock, and immediately begins to move (for example, departure of an ACV from the shore onto ice), the deflection and the rate of its change at the initial time are equal to zero.

We note that, in the derivation the above-mentioned matrix equations, one of the stages was discretization of the plate. In the finite-element method, elements of various types, shapes, and sizes can be used, which allows accounting for a number of features of particular problems (such as curvilinear boundaries of the basin or local weakening of ice). In particular, one of such features is a crack.

To illustrate the method of accounting for cracks in constructing the discrete model, we consider the following example. Let a rectangular plate have a through crack parallel to the side of the plate. In this case, it is reasonable to perform discretization of the plate using joint finite elements with 16 degrees of freedoms [6] (Fig. 1a). To take

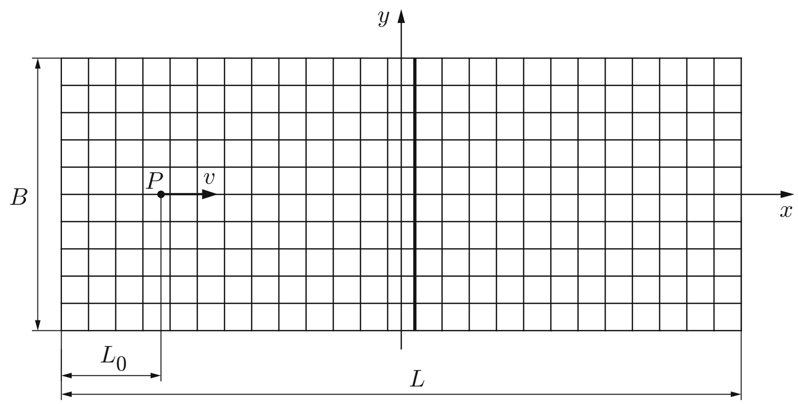

Fig. 2. Discrete model of the plate (the bold line is the crack).

into account the boundary conditions on the crack between its faces, we introduce an additional finite element at the mesh nodes which has the form of a flexible hinge (Fig. 1b). The vertical displacement of the hinge is equal to the plate deflection w at the given node, and the angular displacements q_1 and q_2 are equal to the angular displacements of the crack faces:

$$q_1 = -\frac{\partial w}{\partial x}\Big|_{x=c-0}, \qquad q_2 = -\frac{\partial w}{\partial x}\Big|_{x=c+0}.$$

The angular displacements in the hinge are linked to the moment M that arises in the hinge, which is equal in value and opposite in sign to the corresponding moment on the crack face, by the relation $q_1 - q_2 = AM$, and the stiffness matrix of the hinged finite element is given by

$$K = \begin{bmatrix} 1/A & -1/A \\ -1/A & 1/A \end{bmatrix}.$$

The compliance coefficient A can be determined from full-scale or similar mathematical experiments. However, because full-scale experiments with ice are difficult to perform and because there are advanced elaborate techniques of numerical simulation, the second method is preferred.

It has been shown [7] that, to model a crack by a flexible hinge, it is necessary to know the stiffness per unit length k, which characterizes the bending moment of resistance to the opening of the faces of a crack of unit length. Then, in the stiffness matrix of the flexible hinged element, the coefficient $1/A$ is equal to lk [$l = a$ is the length of the zone of interaction of the crack faces an the corresponding node; a is the length of the side of the finite element (see Fig. 1a)].

As an example, we consider the following problem. An ice plate rectangular in plan and having length $L = 1000$ m and width $B = 400$ m is rigidly clamped on the contour, and the x and y coordinate axes are the symmetry axes of the plate. The crack is on the straight line $x = 20$ m and intersects the plate over the entire width (Fig. 2). At the point $x = -350$ m, $y = 0$ ($L_0 = 150$ m), the plate is acted upon by unit force P, resulting in a corresponding static deflection of the plate. From the state of rest, the force P instantaneously acquires constant velocity v and begins to move on the x axis. The motion of the force results in a dynamic deflection of the plate $w(x, y, t)$. The initial data of the problem are as follows: $E = 0.73 \cdot 10^{10}$ Pa, $\nu = 0.3$, $h = 1$ m, $H = 5$ m, $\rho_i = 900$ kg/m^3, $\rho_w = 1000$ kg/m^3, $\tau_f = 10$ sec, $P = 0.4 \cdot 10^6$ N, and $v = 5$ m/sec. The discrete model of the plate consists of square finite elements with a side length $a = 40$ m; the mesh time step [4] is $\Delta t = 0.2$ sec. The deflections were calculated using a specially designed program, which allow the calculation of the static deflection acquired by the plate prior to the beginning of motion of the force. In this case, the stiffness per unit length k was determined from a mathematical experiment using a wall beam of rectangular cross section with a through crack at the middle of the beam span [8]. During plane bending of the beam, contact stresses arose in the zone of contact of the crack faces and the contact section rotated, i.e., the crack opened through a certain angle. The stresses and displacements in the wall beam were determined using the MSC/NASTRAN software, and the beam was modeled by membrane finite elements. As a result of the calculation, we found the cracks opening angle φ and the bending

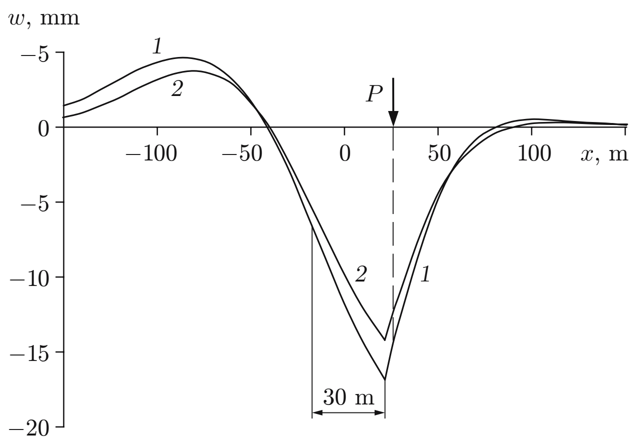

Fig. 3. Deflections of the ice plate on the x at $x = 25$ m and $k = 0$ (1) and $3 \cdot 10^7$ N/rad (2).

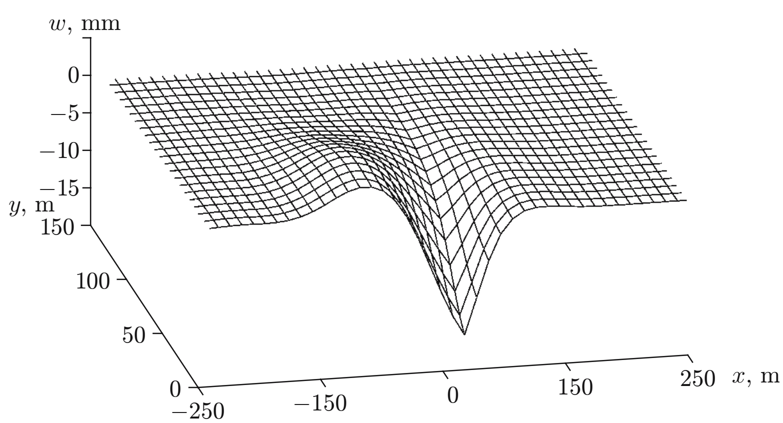

Fig. 4. Deflection of the plate surface.

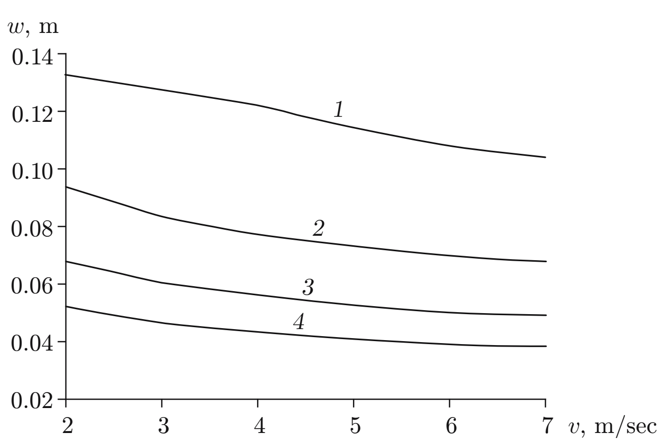

Fig. 5. Maximum deflection versus the velocity of motion of the force for $h = 0.2$ (1), 0.3 (2), 0.4 (3), and 0.5 m (5).

moment M that arises at the beam cross section under the action of the contact stresses. For small deflections, the relation $\varphi(M)$ is almost linear, and it is possible to find the compliance $A = \varphi/M$ and then the coefficient k.

For the indicated parameters of the problem, the calculations were performed with a through straight-line crack modeled by a perfect hinge joint ($k = 0$) and a flexible hinge joint ($k = 3 \cdot 10^7$ N/rad at $h = 1$ m using the method of [8]).

Figure 3 shows the plate deflections at points of the x axis. In the determination of k, the length of the wall beam was set equal to the doubled distance $d = 30$ m from the crack face to the nearest inflection point of the deflection curve on the x axis for $k = 0$. It is evident from Fig. 3 that, in the cases considered, the shape of the elastic surface of the plate is the same; the difference is only in the maximum deflection, and it is 21.4% of the maximum deflection in the case of flexible hinges.

Figure 4 shows the region of the plate surface at $y \geqslant 0$ (because the problem is symmetric about the x axis) at the time when the dynamic deflections take the maximum values (in this case, the force P is at the point $x = 25$ m). In Figs. 3 and 4, one can see that the maximum deflections arise at the points of the plate through which the crack propagates.

To determine the dependence of the maximum dynamic deflections on the velocity of the load, we performed a series of calculations for a rectangular ice plate (see Fig. 2) of length $L = 900$ m and width $B = 300$ m. The through straight-line crack is modeled by perfect hinges, is located on the y axis, and propagates across the entire width of the plate; the discrete model of the plate consists of square finite elements with a side length $a = 25$ m; at the initial time, the point force $P = 0.4 \cdot 10^6$ N is in the state of rest at the point $x = -350$ m, $y = 0$ ($L_0 = 100$ m) on the x axis. The plate has the corresponding static deflection; the load acquires a constant velocity instantaneously at the beginning of motion. The remaining parameters of the problem have the same values as in the previous case. For $h = 0.2$, 0.3, 0.4, and 0.5 m (Fig. 5), the calculations were performed for various load velocities $v = 2$–7 m/sec with a step of 1 m/sec. From Fig. 5, it follows that the maximum deflection decreases as the velocity of the load increases.

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